

# Freudenthal Duality and Generalized Special Geometry

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## ABSTRACT

Freudenthal duality, introduced in [1] and defined as an anti-involution on the dyonic charge vector in  $d = 4$  space-time dimensions for those dualities admitting a quartic invariant, is proved to be a symmetry not only of the classical Bekenstein-Hawking entropy but also of the critical points of the black hole potential.

Furthermore, Freudenthal duality is extended to any generalized special geometry, thus encompassing all  $\mathcal{N} > 2$  supergravities, as well as  $\mathcal{N} = 2$  generic special geometry, not necessarily having a coset space structure.

# 1 Introduction

*Freudenthal duality* was introduced by Borsten, Dahanayake, Duff and Rubens in [1] as an anti-involutive operator on the representation space of the dyonic charge  $\mathcal{Q}^A$ , defined as<sup>1</sup>

$$\widehat{\mathcal{Q}}^A(\mathcal{Q}) \equiv \mathbb{C}^{AB} \frac{\partial \sqrt{|\mathcal{I}_4(\mathcal{Q})|}}{\partial \mathcal{Q}^B}, \quad (1.1)$$

where  $\mathcal{I}_4$  is the invariant polynomial quartic in  $\mathcal{Q}$ , and  $\mathbb{C}$  is the  $2n \times 2n$  symplectic metric ( $\mathbb{C}^T = -\mathbb{C}$ ,  $\mathbb{C}^2 = -\mathbb{I}$ ,  $n$  denoting the number of vector fields of the theory)

$$\mathbb{C} \equiv \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \quad (1.2)$$

Note that  $\widehat{\mathcal{Q}}$  defined by (1.1) is homogeneous of degree one in  $\mathcal{Q}$ .

The basic properties of such an operator are [1]

$$\widehat{\widehat{\mathcal{Q}}} = -\mathcal{Q} \text{ (anti-involution);} \quad (1.3)$$

$$\mathcal{I}_4(\widehat{\widehat{\mathcal{Q}}}) = \mathcal{I}_4(\mathcal{Q}). \quad (1.4)$$

Note that this duality holds for both signs of  $\mathcal{I}_4$ , thus embracing both BPS and non-BPS (“large”) BHs. On the other hand, the definition (1.1) is ill-defined when  $\mathcal{I}_4 = 0$ , corresponding to the so-called “small” BHs, whose Bekenstein-Hawking [2] entropy  $S$  (and thus area of the horizon) vanishes. Also, in the present paper we do not consider the issue of charge quantization and the related restrictions for the implementation of Freudenthal duality [1] (see also remark 1 in Sec. 4).

In mathematical literature, groups with a symplectic representation  $\mathbf{R}$  admitting a completely symmetric rank-4 invariant structure  $\mathbf{q}$  such that the invariant polynomial  $\mathcal{I}_4$  can be defined as<sup>2</sup>

$$\mathcal{I}_4(\mathcal{Q}) \equiv \frac{1}{2} \mathbf{q}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)|_{\mathcal{Q}_1=\mathcal{Q}_2=\mathcal{Q}_3=\mathcal{Q}_4=\mathcal{Q}}, \quad (1.5)$$

are sometimes called “groups of type  $E_7$ ” (see *e.g.* [3]; in the case of  $E_7$ ,  $\mathbf{R}$  is the fundamental representation **56**). “Groups of type  $E_7$ ” are *at least* all the  $U$ -duality groups of  $\mathcal{N} = 2$ ,  $d = 4$  supergravity theories with symmetric vector multiplets’ scalar manifold, as well as of four-dimensional  $\mathcal{N} > 2$ -extended supergravities (all based on symmetric spaces). These include theories (namely, *minimally coupled*  $\mathcal{N} = 2$  [6], and  $\mathcal{N} = 3$  [7] supergravity) in which the basic invariant polynomial is quadratic:  $|\mathcal{I}_2| = \sqrt{|\mathcal{I}_4|}$ , and thus it should replace  $\mathcal{I}_4$  itself in Eqs. (1.1) and (1.4).

In the  $\mathcal{N} = 2$  case, the properties (1.3)-(1.4) are a consequence of the symplectic structure of *special Kähler geometry* endowing vector multiplets’ scalar manifold (see *e.g.* [8], and Refs. therein), while for  $\mathcal{N} > 2$  theories they can be traced back to what one may call *generalized special geometry*. This corresponds to the sigma models which can be consistently coupled to vector fields, as discussed in [9] in general, and then treated in [10, 11] in the context of extended supergravities.

The main result of the present paper is the finding that the Freudenthal duality is in fact a property which can be defined with a dependence on the scalar fields  $\phi$ , parametrizing the whole scalar manifold  $\mathbf{M}$ , in *any generalized special geometry*. Let us recall the definition of the (positive-definite) black hole potential [12]

$$V(\phi, \mathcal{Q}) \equiv -\frac{1}{2} \mathcal{Q}^T \mathcal{M}(\phi) \mathcal{Q}, \quad (1.6)$$

where  $\mathcal{M}$  is the symmetric, real, negative-definite  $2n \times 2n$  matrix made of the vector couplings [13, 8, 12], which depends on the scalar fields  $\phi$  and it is symplectic:

$$\mathcal{M} \mathbb{C} \mathcal{M} = \mathbb{C}. \quad (1.7)$$

<sup>1</sup>For  $\mathcal{I}_4 < 0$ , the definition (1.1) is actually the opposite of definition given in [1]; however, this is immaterial for all subsequent treatment, because  $\phi_H(-\mathcal{Q}) = \phi_H(\mathcal{Q})$  and (3.15) holds.

<sup>2</sup>The normalization of  $\mathbf{q}$  used here is the same one adopted in [1], and it thus differs by a factor 2 with respect to the one adopted *e.g.* in [3], [4] and [5].

In the treatment given below, we will show that  $V$  and  $\mathcal{M}$  are Freudenthal invariant<sup>3</sup> at the attractor points of  $V$  (determining the fixed, attractor [14, 12] values of the scalar fields):

$$V^2|_{\partial V=0}(\phi_H(\mathcal{Q}), \mathcal{Q}) = |\mathcal{I}_4(\mathcal{Q})| = |\mathcal{I}_4(\hat{\mathcal{Q}})|; \quad (1.8)$$

$$\mathcal{M}_H(\hat{\mathcal{Q}}) = \mathcal{M}_H(\mathcal{Q}), \quad (1.9)$$

where  $\mathcal{M}_H(\mathcal{Q}) \equiv \mathcal{M}(\phi_H(\mathcal{Q}))$ .

The results (1.8) and (1.9) will be generalized to a scalar field dependent framework in Sec. 3. Indeed, an interpolating (scalar field dependent) generalization of the Freudenthal duality will be defined in Sec. 3 as a symmetry of the effective 1-dimensional action obtained as reduction of the bosonic sector of the action of an *ungauged*  $\mathcal{N} \geq 2$ ,  $d = 4$  Maxwell-Einstein supergravity by exploiting the space-time symmetries (staticity, spherical symmetry) of the asymptotically flat extremal dyonic black hole background [13, 12].

Remarkably, this reasoning can be extended to non-symmetric  $\mathcal{N} = 2$  special geometries. Within this broad class of theories, definition (1.1) and Eq. (1.8) enjoy the following generalization<sup>4</sup>:

$$\hat{\mathcal{Q}}^A \equiv \mathbb{C}^{AB} \frac{\partial S(\mathcal{Q})}{\partial \mathcal{Q}^B}; \quad (1.10)$$

$$S(\mathcal{Q}) \equiv V|_{\partial V=0}(\phi_H(\mathcal{Q}), \mathcal{Q}). \quad (1.11)$$

$S$  is the classical Bekenstein-Hawking black hole entropy [2], which is homogeneous of degree two in the charges:

$$\mathcal{Q}^A \frac{\partial S(\mathcal{Q})}{\partial \mathcal{Q}^A} = 2S(\mathcal{Q}). \quad (1.12)$$

$\hat{\mathcal{Q}}$  defined by (1.10) is the *symplectic gradient* of  $S$ . Therefore, (1.11) is the statement that  $S$  is invariant when replacing  $\mathcal{Q}$  with the symplectic gradient of  $S$ . Note that in general,  $S$  can be a rather complicated function of  $\mathcal{Q}$  (often not exactly computable, as well), whereas for groups of type  $E_7$  [3],  $S(\mathcal{Q}) = \sqrt{|\mathcal{I}_4(\mathcal{Q})|}$ .

The generalization of Eqs. (1.4) and (1.8) yields, by means of definition (1.10), the result

$$S(\mathcal{Q}) = S\left(\mathbb{C} \frac{\partial S}{\partial \mathcal{Q}}\right). \quad (1.13)$$

This result is general; for instance, it can explicitly be checked to hold in  $\mathcal{N} = 2$ ,  $d = 4$  not symmetric nor homogeneous few-moduli models, such as the ones considered in [15].

These results are consequences of the symplectic structure of (generalized) special geometry, characterized by the fundamental identity [11]

$$\mathcal{M}\mathcal{V}_a = i\mathbb{C}\mathcal{V}_a, \quad (1.14)$$

where  $\mathcal{V}_a(\phi)$  is the  $2n \times 1$  vector of complex symplectic sections, related to  $\mathcal{M}$  through the identity<sup>5</sup> ( $a = 1, \dots, n$ ) [11]

$$\mathcal{M} + i\mathbb{C} = 2(\mathbb{C}\mathcal{V}_a) \left( \bar{\mathcal{V}}^{aT} \mathbb{C} \right). \quad (1.15)$$

In the  $\mathcal{N} = 2$  case, this identity can be rewritten as

$$\mathcal{M} + i\mathbb{C} = 2(\mathbb{C}\mathbf{V}) \left( \bar{\mathbf{V}}^T \mathbb{C} \right) + 2g^{i\bar{j}} \left( \mathbb{C}\bar{\mathbf{V}}_{\bar{j}} \right) \left( \mathbf{V}_i^T \mathbb{C} \right), \quad (1.16)$$

where  $\mathcal{V}_a \equiv (\mathbf{V}, \bar{\mathbf{V}}_{\bar{i}})$ , and  $\mathbf{V} \equiv e^{\mathcal{K}/2} (X^\Lambda, F_\Lambda)^T$  is the  $2n \times 1$  vector of Kähler-covariantly holomorphic sections ( $\mathcal{K}$  denotes the Kähler potential,  $\mathbf{V}_i \equiv D_i \mathbf{V} \equiv (\partial_i + \frac{1}{2} \partial_i \mathcal{K}) \mathbf{V}$ , and  $i = 1, \dots, n_V = n - 1$ , with  $n_V$  standing for the number of vector multiplets).

<sup>3</sup>Throughout our treatment, the subscript “ $H$ ” denotes evaluation at the event horizon of the extremal black hole solution under consideration.

<sup>4</sup>Our definition of  $S(\mathcal{Q})$  is as follows:

$$S(\mathcal{Q}) \equiv V(\phi_H(\mathcal{Q})) \equiv V_H(\mathcal{Q}) = \frac{1}{4\pi} A_H,$$

where in the last step we used the Bekenstein-Hawking entropy-area formula [2] ( $A_H$  denotes the area of the unique event horizon of the extremal BH).

<sup>5</sup>For a specialization to the  $\mathcal{N} = 8$  case, see e.g. [11, 16].

The paper is organized as follows.

In Sec. 2 we consider Freudenthal duality, and we show that, as asserted above, it does not hold only for symmetric scalar manifolds (for which a quartic polynomial  $\mathcal{I}_4$  can be introduced), but rather it can be defined for *all* generalized special geometries, and for all non-degenerate critical points of the effective black hole potential  $V$ , namely for those critical points, BPS or not, for which the supporting charge orbit is “large”, *i.e.*  $S(\mathcal{Q}) \neq 0$ . The case already known in literature [1] is then recovered in Sec. 2.1.

As an example, we consider  $\mathcal{N} = 2$  BPS attractors, for which the sections  $\mathbf{V}_{BPS}$  are given by a complexified charge vector, whose real and imaginary parts are related by Freudenthal duality. This can be traced back to the existence of an *interpolating* (scalar field dependent) Freudenthal duality, which is introduced in Sec. 3, in which the results of Sec. 2 are extended to the whole scalar manifold  $\mathbf{M}$  pertaining to *generalized special geometry*. The effective black hole potential  $V$  (despite being non-invariant under the Freudenthal duality treated in Sec. 2) is shown to be invariant under the *interpolating* Freudenthal duality. It is worth here remarking that the evaluation of the *interpolating* Freudenthal duality at the critical points of  $V$  themselves consistently yields the Freudenthal duality treated in Sec. 2 (which in turn, for groups of type  $E_7$ , reduces to the one originally introduced in [1]).

Given two (static, asymptotically flat, spherically symmetric, dyonic) extremal black hole solutions whose dyonic charge vectors are related by Freudenthal duality, they are characterized by the same critical points of  $V$ , and thus they have the same classical Bekenstein-Hawking entropy. This holds despite the fact that their (attractor) scalar flows are different, and that  $V$  is not invariant under the scalar field independent Freudenthal duality discussed in Sec. 2.

Sec. 4 contains some remarks, in particular concerning the quantization of dyonic vector  $\mathcal{Q}$ , the generalization of the quartic invariant  $\mathcal{I}_4$  to non-symmetric special Kähler geometries, the relation to quaternionic Kähler geometry and its  $\mathcal{N} = 2$ ,  $d = 4$  harmonic superspace formulation.

## 2 Freudenthal Duality

We start by considering the black hole (BH) effective potential [12]

$$V \equiv -\frac{1}{2}\mathcal{Q}^T \mathcal{M} \mathcal{Q}, \quad (2.1)$$

whose criticality conditions read

$$(\partial_i V)_H = (D_i V)_H = 0, \quad \forall i \Leftrightarrow \mathcal{Q}^T (D_i \mathcal{M})_H \mathcal{Q} = 0, \quad \forall i. \quad (2.2)$$

It then follows that

$$\left. \frac{\partial V}{\partial \mathcal{Q}} \right|_H = -\mathcal{M}_H \mathcal{Q} = -\mathcal{M}_H \mathcal{Q} - \frac{1}{2}\mathcal{Q}^T (D_i \mathcal{M})_H \mathcal{Q} \frac{\partial \phi_H^i(\mathcal{Q})}{\partial \mathcal{Q}} = \frac{\partial V_H(\mathcal{Q})}{\partial \mathcal{Q}} = \frac{\partial S(\mathcal{Q})}{\partial \mathcal{Q}}, \quad (2.3)$$

where  $V_H(\mathcal{Q}) \equiv V(\phi_H(\mathcal{Q}), \mathcal{Q})$ ,  $\mathcal{M}_H(\mathcal{Q}) \equiv \mathcal{M}(\phi_H(\mathcal{Q}))$ , and the same for  $(D_i \mathcal{M})_H$ .

As anticipated in the Introduction, the symplectic structure of  $\mathcal{N} \geq 2$ -extended supergravity theories in  $d = 4$  space-time dimensions [8, 11] allows one to introduce a duality operator  $\mathfrak{F}_{\mathcal{Q}}$ , dubbed (after [1]) *Freudenthal duality*, acting on  $Sp(2n, \mathbb{R})$  vectors. Its action on the charge vector  $\mathcal{Q}$  is defined as

$$\mathfrak{F}_{\mathcal{Q}} : \mathcal{Q} \mapsto \hat{\mathcal{Q}}_H \equiv \mathbb{C} \frac{\partial S(\mathcal{Q})}{\partial \mathcal{Q}} = -\mathbb{C} \mathcal{M}(\phi_H(\mathcal{Q})) \mathcal{Q}. \quad (2.4)$$

As a consequence of the result (1.9) and of the symplectic nature of  $\mathcal{M}$  (1.3), the action of  $\mathfrak{F}_{\mathcal{Q}}$  on  $\mathcal{Q}$  is *anti-involutive*:

$$\begin{aligned} \mathfrak{F}_{\mathcal{Q}}^2 &\equiv \mathfrak{F}_{\hat{\mathcal{Q}}_H(\mathcal{Q})} : \mathcal{Q} \mapsto \hat{\hat{\mathcal{Q}}}(\mathcal{Q}) \equiv \hat{\mathcal{Q}}_H(\hat{\mathcal{Q}}_H(\mathcal{Q})) \\ &= \mathbb{C} \mathcal{M}(\phi_H(\hat{\mathcal{Q}}_H(\mathcal{Q}))) \mathbb{C} \mathcal{M}(\phi_H(\mathcal{Q})) \mathcal{Q} = \mathbb{C} \mathcal{M}(\phi_H(\mathcal{Q})) \mathbb{C} \mathcal{M}(\phi_H(\mathcal{Q})) \mathcal{Q} \\ &= -\mathcal{Q}. \end{aligned} \quad (2.5)$$

Note that  $\mathfrak{F}_{\mathcal{Q}}$  acts trivially on the symplectic sections  $\mathcal{V}_{a,H}$  at critical points of  $V$ :

$$\mathfrak{F}_{\mathcal{Q}} : \mathcal{V}_{a,H} \mapsto \mathcal{V}_{a,H}. \quad (2.6)$$

We anticipate that the critical points of the effective BH potential  $V(\phi, \mathcal{Q})$  (defining the attractor configurations of scalar fields at the BH horizon) are also  $\mathfrak{F}_{\mathcal{Q}}$ -invariant, as it will be proved at the end of Sec. 3. An illustrative example is provided by the BPS attractor Eqs. in  $\mathcal{N} = 2$ ,  $d = 4$  special geometry, corresponding to the local vanishing of all *matter charges*

$$(D_i Z)_H \equiv Z_{i,H} = 0 \quad \forall i, \quad (2.7)$$

where  $Z \equiv \mathcal{Q}^T \mathbb{C} \mathbf{V}$  is the complex  $\mathcal{N} = 2$  central charge. By means of (1.16), an equivalent algebraic reformulation of (2.7) reads [17, 18, 19]

$$2i(\bar{Z}\mathbf{V})_{BPS} = \mathcal{Q} + i\mathbb{C}\mathcal{M}_H\mathcal{Q} = \mathcal{Q} - i\hat{\mathcal{Q}}_H. \quad (2.8)$$

Let's now  $\mathfrak{F}_{\mathcal{Q}}$ -transform this equation. By defining the  $1 \times (n_V + 1)$  vector

$$\mathcal{Z}_a \equiv \mathcal{Q}^T \mathbb{C} \mathcal{V}_a \equiv (Z, Z_i), \quad (2.9)$$

and using the general identity (1.14), one can show that

$$\mathfrak{F}_{\mathcal{Q}} : \mathcal{Z}_{a,H} \mapsto \hat{\mathcal{Z}}_{a,H} \equiv \hat{\mathcal{Q}}_H^T \mathbb{C} \mathcal{V}_a = -i\mathcal{Z}_{a,H}. \quad (2.10)$$

By recalling the anti-involutivity property (2.5),  $\mathbf{V}_{H,BPS}$  defined by Eq. (2.8) then turns out to be invariant under  $\mathfrak{F}_{\mathcal{Q}}$ , consistent with (2.6). Therefore, BPS attractor Eqs. (2.8) (and consequently their solutions, corresponding to the non-degenerate BPS critical points of  $V$ ), are  $\mathfrak{F}_{\mathcal{Q}}$ -invariant. By using (2.10), this can also be checked for the equivalent differential conditions (2.7), of course.

As evident from the treatment given below, the criticality conditions of  $V$  (in turn splitting under supersymmetry in the three  $\mathcal{N} = 2$  classes of BPS, non-BPS  $Z_H = 0$  and non-BPS  $Z_H \neq 0$  attractor Eqs.; see *e.g.* [20]) are generally  $\mathfrak{F}_{\mathcal{Q}}$ -invariant, and this result in general holds when generalizing  $\mathfrak{F}_{\mathcal{Q}}$  to  $\mathfrak{F}_{\phi}$  (see Sec. 3), and for in all (generalized) special geometries.

A remarkable consequence of (2.4) and (2.5) is that *the Bekenstein-Hawking BH entropy  $S$  is  $\mathfrak{F}_{\mathcal{Q}}$ -invariant*. Indeed, (2.4) implies that

$$\hat{\mathcal{Q}}_H^T \mathbb{C} \mathcal{Q} = \mathcal{Q} \frac{\partial S}{\partial \mathcal{Q}} = -\mathcal{Q}^T \mathcal{M}_H \mathcal{Q} = 2V_H(\mathcal{Q}) = 2S(\mathcal{Q}); \quad (2.11)$$

$$\Updownarrow$$

$$S(\mathcal{Q}) = \frac{1}{2} \langle \hat{\mathcal{Q}}_H, \mathcal{Q} \rangle \Rightarrow \hat{S}(\mathcal{Q}) \equiv S(\hat{\mathcal{Q}}_H) = S(\mathcal{Q}). \quad (2.12)$$

Note that (2.11) is consistent with the fact that  $S(\mathcal{Q}) = V(\phi_H(\mathcal{Q}), \mathcal{Q}) \equiv V_H(\mathcal{Q})$  is homogeneous of degree two (and thus even) in the charges  $\mathcal{Q}$ .

(2.12) is a general result, which holds in any ( $\mathcal{N} \geq 2$ ,  $d = 4$ ) generalized special geometry. Within this broad class of theories, any two BH solutions whose dyonic charge vectors are related by  $\mathfrak{F}_{\mathcal{Q}}$  have the same critical points of  $V$  (as it will be proved at the end of Sec. 3) and the same classical entropy  $S$ . This holds despite the fact that  $V(\phi, \mathcal{Q}) \neq V(\phi, \hat{\mathcal{Q}}_H)$  (on the other hand, Eqs. (2.11) and (2.12) imply that  $V_H(\mathcal{Q}) \neq V_H(\hat{\mathcal{Q}}_H)$ ). It is worth here noting that the action of  $\mathfrak{F}_{\mathcal{Q}}$  on rank-4 classical charge orbits preserves both the supersymmetry properties and the rank <sup>6</sup> of these orbits.

## 2.1 The case of Groups of Type $E_7$

For those generalized special geometries [11] related to groups of type  $E_7$  [3], which are endowed with an invariant structure  $\mathbf{q}$  defining the quartic invariant  $\mathcal{I}_4$  (1.5),  $\mathfrak{F}_{\mathcal{Q}}$  reduces to the *non-polynomial* Freudenthal duality introduced in [1], here denoted by  $\mathfrak{F}_{\mathcal{Q},\mathbf{q}}$  (recall definition (1.1) and Footnote 1):

$$\mathfrak{F}_{\mathcal{Q},\mathbf{q}} : \mathcal{Q} \mapsto \hat{\mathcal{Q}}_{H,\mathbf{q}}(\mathcal{Q}) \equiv -\mathbb{C}\mathcal{M}_{H,\mathbf{q}}\mathcal{Q} = \mathbb{C} \frac{\partial \sqrt{|\mathcal{I}_4|}}{\partial \mathcal{Q}} = \epsilon \frac{1}{2\sqrt{|\mathcal{I}_4|}} \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}}, \quad (2.13)$$

where  $\mathcal{I}_4 \equiv \epsilon |\mathcal{I}_4|$ .

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<sup>6</sup>Consistent with [1], here by *rank* we mean the rank of any representative  $\mathcal{Q}$  of the orbit as element of the corresponding Freudenthal triple system, as defined in [21].

From points  $a)$ ,  $e)$ ,  $d)$  and  $g)$  of Lemma 11 of [3], various results on all possible evaluations of  $\mathbf{q}$  on  $\mathcal{Q}$  and  $\widehat{\mathcal{Q}}_{H,\mathbf{q}}$  follow:

$$\mathbf{q}\left(\mathcal{Q}^3\widehat{\mathcal{Q}}_{H,\mathbf{q}}\right) = 0 = \mathbf{q}\left(\mathcal{Q}\widehat{\mathcal{Q}}_{H,\mathbf{q}}^3\right); \quad (2.14)$$

$$\mathbf{q}\left(\mathcal{Q}^2\widehat{\mathcal{Q}}_{H,\mathbf{q}}^2\right) = \frac{1}{3}\mathbf{q}\left(\mathcal{Q}^4\right), \quad (2.15)$$

and

$$\mathbf{q}\left(\widehat{\mathcal{Q}}_{H,\mathbf{q}}^4\right) = \mathbf{q}\left(\mathcal{Q}^4\right) \stackrel{(1,5)}{\Leftrightarrow} \mathfrak{F}_{\mathcal{Q},\mathbf{q}}\left(\mathcal{I}_4\left(\mathcal{Q}^4\right)\right) \equiv \mathcal{I}_4\left(\widehat{\mathcal{Q}}_{H,\mathbf{q}}^4\right) = \mathcal{I}_4\left(\mathcal{Q}^4\right). \quad (2.16)$$

(2.16) consistently expresses the  $\mathfrak{F}_{\mathcal{Q},\mathbf{q}}$ -invariance of the  $U$ -invariant polynomial  $\mathcal{I}_4$  defined by (1.5).

### 3 Interpolating (Field-Dependent) Formulation

As a consequence of the basic identities of generalized special geometry [11], one can introduce an alternative operator  $\mathfrak{F}_\phi$ , which is *scalar-dependent* and it is defined *all over the scalar manifold*  $\mathbf{M}$ . The action of  $\mathfrak{F}_\phi$  on the charge vector  $\mathcal{Q}$  and on the symplectic sections  $\mathcal{V}_a$  is respectively defined as

$$\mathfrak{F}_\phi : \mathcal{Q} \mapsto \widehat{\mathcal{Q}}_\phi(\phi) \equiv \mathbb{C} \frac{\partial V(\phi, \mathcal{Q})}{\partial \mathcal{Q}} = -\mathbb{C} \mathcal{M}(\phi) \mathcal{Q}; \quad (3.1)$$

$$\mathfrak{F}_\phi : \mathcal{V}_a \mapsto \mathcal{V}_a. \quad (3.2)$$

Note that (3.2) implies that the scalar fields  $\phi$ , coordinatizing  $\mathbf{M}$ , are trivially  $\mathfrak{F}_\phi$ -invariant.

As a consequence of the symplectic nature (1.3) of  $\mathcal{M}(\phi)$ , it is immediate to see that the non-trivial action of  $\mathfrak{F}_\phi$  on  $\mathcal{Q}$  is *anti-involutive*:

$$\mathfrak{F}_\phi^2 : \mathcal{Q} \mapsto \widehat{\widehat{\mathcal{Q}}}_\phi(\phi) = -\mathcal{Q}. \quad (3.3)$$

Note that (2.4) and (2.6) are nothing but (3.1) and (3.2) evaluated at the critical points of  $V$ , respectively. Furthermore, the general identity (1.15) and the trivial action (3.2) of  $\mathfrak{F}_\phi$  on the symplectic sections  $\mathcal{V}_a$  imply the matrix  $\mathcal{M}(\phi)$  to be  $\mathfrak{F}_\phi$ -invariant:

$$\widehat{\mathcal{M}}_\phi(\phi) \equiv \mathcal{M}(\mathfrak{F}_\phi(\mathcal{V}_a(\phi))) = \mathcal{M}(\phi). \quad (3.4)$$

At the critical points of  $V$ , Eq. (3.4) reduces to Eq. (1.9).

As anticipated, a consequence of (3.1) and (3.3), which can ultimately be traced back to the basic properties of generalized special geometry, is that *the effective BH potential  $V(\phi, \mathcal{Q})$  is  $\mathfrak{F}_\phi$ -invariant*. Indeed, (3.1) implies that

$$\widehat{\mathcal{Q}}_\phi^T \mathbb{C} \mathcal{Q} = -\mathcal{Q}^T \mathcal{M}(\phi) \mathcal{Q} = 2V(\phi, \mathcal{Q}); \quad (3.5)$$

$$\begin{aligned} & \Downarrow \\ V(\phi, \mathcal{Q}) &= \frac{1}{2} \left\langle \widehat{\mathcal{Q}}_\phi, \mathcal{Q} \right\rangle; \end{aligned} \quad (3.6)$$

$$\begin{aligned} & \Downarrow \\ \widehat{V}_\phi(\phi, \mathcal{Q}) &\equiv V\left(\phi, \widehat{\mathcal{Q}}_\phi\right) = \frac{1}{2} \widehat{\mathcal{Q}}_\phi^T \mathbb{C} \widehat{\mathcal{Q}}_\phi = \frac{1}{2} \widehat{\mathcal{Q}}_\phi^T \mathbb{C} \mathcal{Q} = V(\phi, \mathcal{Q}). \end{aligned} \quad (3.7)$$

Note that (3.5) is consistent with the fact that  $V$  is a polynomial homogeneous of degree two (and thus even) in the charges  $\mathcal{Q}$ . Note that (3.15) and (3.7) immediately imply that

$$V_H(\mathcal{Q}) \equiv V_H(\phi_H(\mathcal{Q}), \mathcal{Q}) = V_H\left(\phi_H\left(\widehat{\mathcal{Q}}_H\right), \mathcal{Q}\right) \equiv V_H\left(\widehat{\mathcal{Q}}_H(\mathcal{Q})\right) = \frac{1}{2} \left\langle \widehat{\mathcal{Q}}_H(\mathcal{Q}), \mathcal{Q} \right\rangle. \quad (3.8)$$

(3.7) is a general result, which holds in any ( $\mathcal{N} \geq 2$ ,  $d = 4$ ) symplectic geometry.

Let us now prove that the critical points of  $V(\phi, \mathcal{Q})$  coincide with the critical points of  $V\left(\phi, \widehat{\mathcal{Q}}_H(\mathcal{Q}) \equiv \mathfrak{F}_\mathcal{Q}(\mathcal{Q})\right)$ , namely that the critical points  $\phi_H(\mathcal{Q})$  of  $V(\phi, \mathcal{Q})$  are  $\mathfrak{F}_\mathcal{Q}$ -invariant.

First, we observe that by differentiating the general identity (1.7) with respect to the scalar fields  $\phi$  and using (1.7) again, one obtains

$$\frac{\partial \mathcal{M}(\phi)}{\partial \phi} = \mathcal{M}(\phi) \mathbb{C} \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \mathbb{C} \mathcal{M}(\phi), \quad (3.9)$$

which multiplied by  $-\frac{1}{2}\mathcal{Q}^T$  on the left and by  $\mathcal{Q}$  on the right yields, by recalling definitions (1.6) and (3.1):

$$\frac{\partial V(\phi, \mathcal{Q})}{\partial \phi} = -\frac{1}{2}\mathcal{Q}^T \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \mathcal{Q} = \frac{1}{2}\widehat{\mathcal{Q}}_\phi^T \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \widehat{\mathcal{Q}}_\phi. \quad (3.10)$$

Therefore, the results (3.7) and (3.10) imply

$$\frac{\partial V(\phi, \mathcal{Q})}{\partial \phi} = \frac{\partial V(\phi, \widehat{\mathcal{Q}}_\phi)}{\partial \phi}, \quad (3.11)$$

whose evaluation at  $\phi_H(\mathcal{Q})$  yields by definition:

$$0 = \left. \frac{\partial V(\phi, \mathcal{Q})}{\partial \phi} \right|_{\phi_H(\mathcal{Q})} = \frac{1}{2}\widehat{\mathcal{Q}}_{\phi_H(\mathcal{Q})}^T \left. \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \right|_{\phi_H(\mathcal{Q})} \widehat{\mathcal{Q}}_{\phi_H(\mathcal{Q})} \equiv \frac{1}{2}\widehat{\mathcal{Q}}_H^T \left. \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \right|_{\phi_H(\mathcal{Q})} \widehat{\mathcal{Q}}_H. \quad (3.12)$$

As a final step, let us consider

$$V(\phi, \widehat{\mathcal{Q}}_H) \equiv -\frac{1}{2}\widehat{\mathcal{Q}}_H^T \mathcal{M}(\phi) \widehat{\mathcal{Q}}_H, \quad (3.13)$$

whose derivative with respect to scalar fields reads, evaluated at  $\phi_H(\widehat{\mathcal{Q}}_H) \equiv \widehat{\phi}_H(\mathcal{Q})$  reads by definition

$$0 \equiv \left. \frac{\partial V(\phi, \widehat{\mathcal{Q}}_H)}{\partial \phi} \right|_{\phi_H(\widehat{\mathcal{Q}}_H)} = -\frac{1}{2}\widehat{\mathcal{Q}}_H^T \left. \frac{\partial \mathcal{M}(\phi)}{\partial \phi} \right|_{\phi_H(\widehat{\mathcal{Q}}_H)} \widehat{\mathcal{Q}}_H. \quad (3.14)$$

The comparison of (3.12) and (3.14) implies that both  $\phi_H(\mathcal{Q})$  and  $\phi_H(\widehat{\mathcal{Q}}_H)$  are critical points of  $V$ . On the other hand, in all  $\mathcal{N} \geq 2$ ,  $d = 4$  extended supergravities (vector multiplets') scalar fields can be parametrised projectively in terms of symplectic sections  $\mathcal{V}_a$ . Thus, the  $\mathfrak{F}_\mathcal{Q}$ -invariance of  $\mathcal{V}_{a,H}$  expressed by (2.6) implies the  $\mathfrak{F}_\mathcal{Q}$ -invariance of the critical points  $\phi_H(\mathcal{Q})$  of the effective BH potential  $V(\phi, \mathcal{Q})$ :

$$\mathfrak{F}_\mathcal{Q}(\phi_H(\mathcal{Q})) \equiv \phi_H(\mathfrak{F}_\mathcal{Q}(\mathcal{Q})) = \phi_H(\mathcal{Q}) \quad \blacksquare \quad (3.15)$$

Thus, as anticipated, two  $\mathfrak{F}_\mathcal{Q}$ -dual BH solutions (namely, two BH solutions whose dyonic charge vectors are related by the scalar field-independent Freudenthal duality  $\mathfrak{F}_\mathcal{Q}$ ) have different *off-shell* effective BH potentials (but with the same critical points) and same entropy  $S$ , despite the fact their (attractor) scalar flows are generally different.

As mentioned above, it is immediate to check that all definitions and results concerning  $\mathfrak{F}_\phi$  consistently reduce, along the geometrical *locus* in  $\mathbf{M}$  defined by the (non-degenerate) critical points of  $V$ , to the analogous definitions and results for  $\mathfrak{F}_\mathcal{Q}$  considered in Sec. 2. Namely:

$$\mathfrak{F}_\phi|_{\partial V(\phi, \mathcal{Q})=0, V \neq 0} \equiv \mathfrak{F}_{\phi_H(\mathcal{Q})} = \mathfrak{F}_\mathcal{Q}, \quad (3.16)$$

and thus:

$$\widehat{\mathcal{Q}}_H(\mathcal{Q}) = \widehat{\mathcal{Q}}_{\phi_H(\mathcal{Q})}. \quad (3.17)$$

## 4 Final Remarks

### 1. On charge quantization

The algebraic nature of the  $\mathcal{N} = 2$  BPS Attractor Eqs. (2.8) sheds light on the relevance of Freudenthal duality in presence of Dirac-Schwinger-Zwanzinger dyonic charge quantization conditions, namely in the case in which  $\mathcal{Q}$  is integer. Indeed, by requiring (along the lines of the analysis of [1])  $\widehat{\mathcal{Q}}_H$  to be an integer as well,

it follows that (within a suitably Kähler gauge-fixing in which  $2i\bar{Z} \equiv 1$ )  $\mathbf{V}_{BPS}$  is a *complex integer* (namely,  $\mathbf{V}_{BPS} \in \mathbb{Z} + i\mathbb{Z}$ ).

## 2. On the generalization of $\mathcal{I}_4$

The rank-4 completely symmetric invariant structure  $\mathbf{q}$  (1.5) characterizing all “groups of type  $E_7$ ” [3] (whose suitable real forms are the  $U$ -duality groups of  $d = 4$  supergravity theories with symmetric scalar manifolds) has a generalization to the case of arbitrary  $\mathcal{N} = 2$  *non-symmetric* vector multiplets’ special Kähler scalar manifolds  $\mathbf{M}$ . This is obtained by considering the function  $\mathcal{I}_{4,\mathcal{N}=2,symm}$  (given by Eq. (5.36) of [22]):

$$\mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{Q}) = \left(|Z|^2 - Z_i \bar{Z}^i\right)^2 + \frac{2}{3}i \left(Z N_3(\bar{Z}) - \bar{Z} \bar{N}_3(Z)\right) - g^{i\bar{j}} C_{ijk} \bar{C}_{i\bar{l}\bar{m}} \bar{Z}^j \bar{Z}^k Z^l Z^{\bar{m}}, \quad (4.1)$$

where  $N_3(\bar{Z}) \equiv C_{ijk} \bar{Z}^i \bar{Z}^j \bar{Z}^k$ , with  $C_{ijk}$  denoting the  $C$ -tensor of special Kähler geometry.  $\mathcal{I}_{4,\mathcal{N}=2,symm}$  is an homogeneous polynomial in  $\mathcal{Q}$ ; note also that (4.1) is independent on the choice of the symplectic frame and also manifestly invariant under diffeomorphisms in  $\mathbf{M}$ .

As stated in [22],  $\mathcal{I}_{4,\mathcal{N}=2,symm}$  is independent on  $\phi$  *at least* for symmetric  $\mathbf{M}$ , whereas generally it is  $\phi$ -dependent; its fourth derivative with respect to charges defines the following rank-4 completely symmetric ( $\mathcal{Q}$ -independent) tensor

$$\Omega_{ABCD}(\phi) \equiv \frac{1}{12} \frac{\partial^4 \mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{Q})}{\partial \mathcal{Q}^A \partial \mathcal{Q}^B \partial \mathcal{Q}^C \partial \mathcal{Q}^D} \Leftrightarrow \mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{Q}) = \frac{1}{2} \Omega_{ABCD}(\phi) \mathcal{Q}^A \mathcal{Q}^B \mathcal{Q}^C \mathcal{Q}^D. \quad (4.2)$$

For *symmetric* special Kähler manifolds  $\Omega_{ABCD} = \mathbf{q}_{ABCD}$  defined by (1.5) is an invariant structure of the charge representation  $\mathcal{Q}$  of “groups of type  $E_7$ ” [3], and thus  $\mathcal{I}_{4,\mathcal{N}=2,symm} = \mathcal{I}_4(\mathcal{Q})$ .

It is here worth recalling that the general relation between  $\mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{Q})$  (4.1) and the square of the effective BH potential (1.6) [12]

$$V^2(\phi, \mathcal{Q}) = \frac{1}{4} \mathcal{M}_{(AB}(\phi) \mathcal{M}_{CD)}(\phi) \mathcal{Q}^A \mathcal{Q}^B \mathcal{Q}^C \mathcal{Q}^D = \left(|Z|^2 + Z_i \bar{Z}^i\right)^2 \quad (4.3)$$

enjoys the general expression at the various classes of (non-degenerate) critical points of  $V$  itself in a *generic*  $\mathcal{N} = 2$  special Kähler geometry:

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_H = V_H^2 - \frac{32}{3} |Z|_H^2 \left(Z_i \bar{Z}^i\right)_H. \quad (4.4)$$

Namely:

- at BPS critical points, defined by (2.7), it holds that [22]

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_H = |Z|_H^4 = V_H^2; \quad (4.5)$$

- at those non-BPS critical points defined by  $Z_H = 0$ , it holds that [22]

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_H = \left(Z_i \bar{Z}^i\right)_H^2 = \left[g^{i\bar{j}} (\partial_i Z) \bar{\partial}_{\bar{j}} \bar{Z}\right]_H^2 = V_H^2; \quad (4.6)$$

- at the general class of non-BPS critical points with  $Z_H \neq 0$ , the relation (4.4) holds, simplifying to

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_H = -16 |Z|_H^4 = -V_H^2 \quad (4.7)$$

when  $\left(Z_i \bar{Z}^i\right)_H = 3 |Z|_H^2$ , as *e.g.* it holds for *symmetric* special Kähler spaces.

## 3. On the relation to special quaternionic geometry

As noticed in [22],  $\mathcal{I}_{4,\mathcal{N}=2,symm}$  is remarkably related to the *geodesic potential* defined in the  $d = 4 \rightarrow 3$  dimensional reduction of the considered  $\mathcal{N} = 2$  theory. Under such a reduction, the  $d = 4$  vector multiplets’ special Kähler manifold  $\mathbf{M}$  ( $\dim_{\mathbb{C}} = n_V$ ) enlarges to a *special* quaternionic Kähler manifold  $\mathfrak{M}$  ( $\dim_{\mathbb{H}} = n_V + 1$ ) given by  $c$ -map [23, 24] of  $\mathbf{M}$  itself :  $\mathfrak{M} = c(\mathbf{M})$ . By specifying Eq. (5.36) of [22] in the *special coordinates*’ symplectic frame,  $\mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{Q})$  matches the opposite of the function  $h$  defined by Eq. (4.42) of [25], within the analysis of *special* quaternionic Kähler geometry.



This relation can be strengthened by observing that the tensor  $\Omega_{ABCD}$  defined by (4.2) is proportional to the  $\Omega$ -tensor of quaternionic geometry, related to the Riemann tensor by Eq. (15) of [26].

#### 4. On the relation to $\mathcal{N} = 2$ , $d = 4$ harmonic superspace

For *symmetric*  $\mathbf{M}$ , it holds that

$$\mathcal{I}_{4,\mathcal{N}=2,symm} = \mathcal{I}_4(\mathcal{Q}) = \frac{1}{2}\mathbf{q}(\mathcal{Q}^4) = \mathcal{L}^{+4}, \quad (4.8)$$

where  $\mathcal{L}^{+4}$  is the *quaternionic potential* of symmetric quaternionic Kähler non-compact spaces [27, 28], determined in [29, 30] (see also [31]) with  $\mathcal{N} = 2$ ,  $d = 4$  harmonic superspace techniques, in which the  $d = 4$  BH dyonic charge vector  $\mathcal{Q}$  becomes the  $SU(2)$  harmonic coordinate  $Q^+$ . Note that for the  $c$ -map of the non-compact  $\mathbb{CP}^n$  special Kähler manifold, given by the special quaternionic Kähler manifold

$$c(\mathbb{CP}^n) = \frac{SU(2, n+1)}{SU(2) \times SU(n+1) \times U(1)}, \quad (4.9)$$

$\mathcal{L}^{+4}$  becomes a perfect square (see Eq. (11.5) of [30]), consistent with the observation below Eq. (1.4). When considering *special* quaternionic Kähler spaces (*i.e.* the images of special Kähler spaces through  $c$ -map) [23, 24], the coordinates  $\mathcal{Q}$  of  $\mathfrak{M}$  come from the  $d = 3$  dualization of the  $n = n_V + 1$   $d = 4$  vector fields. Note that, consistent with (4.8), the  $\Omega$ -tensor of special quaternionic geometry never vanishes [24]. Moreover, all symmetric non-compact quaternionic spaces are *special* quaternionic, with the exception of the *quaternionic projective spaces* [27]

$$\mathbb{HP}^n \equiv \frac{USp(2, 2n)}{USp(2) \times USp(2n)}, \quad (4.10)$$

which are also the unique example of symmetric quaternionic space with vanishing  $\Omega$ -tensor and thus vanishing  $\mathcal{L}^{+4}$  (see Sec. 8 of [30]).

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